EISENSTEIN SERIES ASSOCIATED WITH $\Gamma_0(2)$

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Abstract. In this paper, we define three normalized Eisenstein series $\mathcal{P}$, e, and $\mathcal{Q}$ associated with $\Gamma_0(2)$, and derive three differntial equations satisfied by them by using some trigonometirc identities. By using these three formulas, we define a differential equation depending on the weights of modular forms on $\Gamma_0(2)$ and then construct their modular solutions by using orthogonal polynomials and Gaussian hypergeometric series. We also construct a certain class of infinite series connected with the triangular numbers. Finally, we derive a combinatorial identity from an infinite series involving the triangular numbers.

1. Introduction

In his notation [13], Ramanujan's three primary Eisenstein series are defined for $|q| < 1$ by

$$P := P(q) = 1 - 24\Phi_{0,1}(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{1.1}$$

$$Q := Q(q) = 1 + 240\Phi_{0,3}(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{1.2}$$

$$R := R(q) = 1 - 504\Phi_{0,5}(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}, \tag{1.3}$$

where

$$\Phi_{r,s} := \Phi_{r,s}(q) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s q^{mn}$$

for integers $r, s \geq 0$.

In more contemporary notation, the normalized Eisenstein series associated with the full modular group $SL_2(\mathbb{Z})$ are defined, for each even integer $k \geq 4$, by

$$E_k(z) = \frac{1}{2}\sum (cz+d)^{-k},$$

where the summation is over all coprime pairs of integers c and d, and Im $z > 0$. Then it is known that $E_k(z)$ has the Fourier expansion [9]

$$E_k := E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{1.4}$$

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where $q = e^{2\pi i z}$, B_k is the kth Bernoulli number and

$$\sigma_k(n) := \sum_{d|n} d^k. \tag{1.5}$$

As usual, we set $\sigma_1(n) = \sigma(n)$ and $\sigma_k(n) = 0$ if $n \notin \mathbb{N}$. Note that $E_4(z) = Q(q)$ and $E_6(z) = R(q)$ are holomorphic modular forms on the full modular group $SL_2(\mathbb{Z})$ of weights 4 and 6, respectively [9, p. 109]. It is well–known that $E_2(z) = P(q)$ is not a modular from of weight 2 [9, p. 12], called a *quasi–modular* form. The Eisenstein series (1.1)–(1.3) satisfy the differential equations [13, eq. (30)], [14, p. 142]

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \tag{1.6}$$

$$q\frac{dQ}{dq} = \frac{PQ - R}{3}, \tag{1.7}$$

$$q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{1.8}$$

In connection with Ramanujan's functions $\Phi_{r,s}$, V. Ramamani, in her paper [12, pp. 279–286], defined analogues of the Ramanujan functions $\Phi_{r,s}$ for integers $r, s \geq 0$, by

$$\Psi_{r,s} := \Psi_{r,s}(q) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}m^r n^s q^{mn}. \tag{1.9}$$

In contrast to $\Phi_{r,s}$, the functions $\Psi_{r,s}$ are not symmetric in r and s. In connection with $\Psi_{r,s}$, let us define three functions $\mathcal{P}$, e, and $\mathcal{Q}$ by

$$\mathcal{P} := \mathcal{P}(q) = 1 + 8\Psi_{0,1}(q) = 1 + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n}, \tag{1.10}$$

$$e := e(q) = 1 + 24\Psi_{1,0}(q) = 1 + 24\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n}, \tag{1.11}$$

$$\mathcal{Q} := \mathcal{Q}(q) = 1 - 16\Psi_{0,3}(q) = 1 - 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^3q^n}{1-q^n}. \tag{1.12}$$

Using the theory of elliptic functions, Ramamani [11] proved that when $r + s$ is odd, $\Psi_{r,s}$ can be expressed as a polynomial in $\mathcal{P}$, e, and $\mathcal{Q}$. We remark that for $r + s$ odd, the function $\Psi_{r,s}$ is related to the normalized Eisenstein series on $\Gamma_0(2)$, where the modular subgroup $\Gamma_0(2)$ is defined by

$$\Gamma_0(2) := \left\{\gamma \in SL_2(\mathbb{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod 2\right\}. \tag{1.13}$$

The normalized Eisenstein series associated with $\Gamma_0(2)$ are defined, for even integer $k \geq 4$, by

$$\mathcal{E}_k := \mathcal{E}_k(z) = 1 - \frac{2k}{(1-2^k)B_k}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^{k-1}q^n}{1-q^n}, \quad q := e^{2\pi i z}, \quad \operatorname{Im} z > 0. \tag{1.14}$$

Then the series $\mathcal{E}_k(z)$ are modular forms of weight k on $\Gamma_0(2)$ which vanish at the cusp zero [4, Theorem 1.1]. It is clear that $\mathcal{E}_4(z) = \mathcal{Q}(q)$ is the relevant modular form of weight 4 on $\Gamma_0(2)$. When $k = 2$, it turns out that $\mathcal{E}_2(z) = \mathcal{P}(q)$ is not a modular form on this group (see (2.36)), but it plays important roles in the theory of modular forms of level 2. Note that the function $e(q)$ is indeed the modular form of weight 2 on $\Gamma_0(2)$ [2, Lemma 3.3].

In Section 2, we derive relations for $\Psi_{r,s}$, for odd $r+s$, from trigonometric identities [13, eqs. (17), (18)] and then, in the same manner as (1.6)–(1.8) are proved, we obtain the differential equations

$$q\frac{d\mathcal{P}}{dq} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4}, \tag{1.15}$$

$$q\frac{de}{dq} = \frac{e\mathcal{P} - \mathcal{Q}}{2}, \tag{1.16}$$

$$q\frac{d\mathcal{Q}}{dq} = \mathcal{P}\mathcal{Q} - e\mathcal{Q}. \tag{1.17}$$

The proofs will be given in Theorem 2.2 and Theorem 2.4. At the end of Section 2, we mention an alternative proof of these formulas using the theory of modular forms. In Section 3, by using these formulas (1.15)–(1.17), we define a differential equation depending on the weight k of modular forms on $\Gamma_0(2)$ and then construct its modular solutions by using orthogonal polynomials. We also find the hypergeometric structure for the solutions of this differential equation. In section 4, we construct a certain class of infinite series connected with the triangular numbers. Finally, we derive a combinatorial interpretation from one of identities of infinite series which we construct in Section 4.

2. Differential equations for $\mathcal{P}$, e, and $\mathcal{Q}$

Ramamani [11] proved that for odd $s \geq 3$, $\Psi_{0,s}$ can be expressed as a polynomial in e and $\mathcal{Q}$ by using the theory of elliptic functions. We can observe this by comparing the coefficients of θ^n in the trigonometric identity [13, eq. (18)]

$$\begin{aligned}\Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\Big)^2 &+ \frac{1}{12}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}(5+\cos n\theta) \\ &= \Big(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12} + \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(1-\cos n\theta)\Big)^2.\end{aligned} \tag{2.1}$$

After replacing θ by $\pi+\theta$ in (2.1), let us expand $\tan\theta$ and $\cos n\theta$ in their Taylor series about 0 for each $n = 1, 2, \ldots$. We therefore find that

$$\begin{aligned}\frac{1}{144} + \frac{\theta^2}{192} + \frac{17\theta^4}{9216} + \cdots + \frac{1}{12}\Bigg\{&\Big(4\frac{q}{1-q} + 6\frac{2^3q^2}{1-q^2} + 4\frac{3^3q^3}{1-q^3} + \cdots\Big) \\ &+ \frac{1}{2!}\Big(\frac{q}{1-q} - \frac{2^5q^2}{1-q^2} + \frac{3^5q^3}{1-q^3} - \cdots\Big)\theta^2\end{aligned}$$

$$- \frac{1}{4!}\Big(\frac{q}{1-q} - \frac{2^7q^2}{1-q^2} + \frac{3^7q^3}{1-q^3} - \cdots\Big)\theta^4 + \cdots \Bigg\}$$

$$(2.2) \qquad = \Bigg\{ \frac{1}{12} + 2\Big(\frac{q}{1-q} + \frac{3q^3}{1-q^3} + \frac{5q^5}{1-q^5} + \cdots\Big) + \frac{1}{2!}\Big(\frac{1}{16} - \frac{q}{1-q} + \frac{2^3q^2}{1-q^2} - \frac{3^3q^3}{1-q^3} + \cdots\Big)\theta^2 + \frac{1}{4!}\Big(\frac{1}{8} + \frac{q}{1-q} - \frac{2^5q^2}{1-q^2} + \frac{3^5q^3}{1-q^3} - \cdots\Big)\theta^4 + \cdots \Bigg\}^2.$$

Before going further, we need to introduce an alternative representation for $e(q)$. By the elementary fact

$$\frac{x}{1+x} = \frac{x}{1-x} - \frac{2x^2}{1-x^2},$$

we find that

$$\begin{aligned} e(q) &= 1 + 24\sum_{n=1}^{\infty} \frac{nq^n}{1+q^n} \\ &= 1 + 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 24\sum_{n=1}^{\infty} \frac{2nq^{2n}}{1-q^{2n}} \\ &= 1 + 24\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}. \qquad (2.3) \end{aligned}$$

So we can rewrite (2.2) as

$$\begin{aligned} &\frac{1}{144} + \frac{1}{6}\left(2\frac{q}{1-q} + 3\frac{2^3q^2}{1-q^2} + 2\frac{3^3q^3}{1-q^3} + \cdots\right) + \left(\frac{1}{192} + \frac{\Psi_{0,5}}{12\cdot 2!}\right)\theta^2 \\ &\quad + \left(\frac{17}{9216} - \frac{\Psi_{0,7}}{12\cdot 4!}\right)\theta^4 + \left(\frac{31}{69120} + \frac{\Psi_{0,9}}{12\cdot 6!}\right)\theta^6 + \cdots \\ &= \left\{\frac{1}{12} + \frac{e-1}{12} + \frac{\mathcal{Q}}{16\cdot 2!}\theta^2 + \frac{1+8\Psi_{0,5}}{8\cdot 4!}\theta^4 + \cdots\right\}^2 \\ &= \frac{e^2}{144} + \frac{e\mathcal{Q}}{192}\theta^2 + \left(\frac{\mathcal{Q}^2}{1024} + \frac{e(1+8\Psi_{0,5})}{1152}\right)\theta^4 \qquad (2.4) \\ &\quad + \left(\frac{e(17-32\Psi_{0,7})}{139240} + \frac{\mathcal{Q}(1+8\Psi_{0,5})}{3072}\right)\theta^6 + \cdots . \end{aligned}$$

So if we compare the coefficients of θ^2 on both sides of (2.4), then we have

$$1 + 8\Psi_{0,5} = e\mathcal{Q}. \qquad (2.5)$$

Similarly, equating coefficients of θ^4 and using (2.5), we obtain the identity

$$17-32\Psi_{0,7}=9\mathcal{Q}^2+8e(1+8\Psi_{0,5})=9\mathcal{Q}^2+8e^2\mathcal{Q}.$$

Successively comparing the coefficients of $\theta^n,\ n=6,10,12,\dots$ on both sides, we easily obtain the following theorem.

Theorem 2.1. *For even integer $k\geq 4$,*

$$\mathcal{E}_k=\sum_{\substack{2m+4n=k\\ m\geq 0,n\geq 1}}\alpha_{m,n}e^m\mathcal{Q}^n, \tag{2.6}$$

where $\alpha_{m,n}$ are constants.

The first few examples of Theorem 2.1 are the relations contained in the following Table I.

$$1-16\Psi_{0,3}=\mathcal{Q}, \tag{2.7}$$

$$1+8\Psi_{0,5}=e\mathcal{Q}, \tag{2.8}$$

$$17-32\Psi_{0,7}=8e^2\mathcal{Q}+9\mathcal{Q}^2, \tag{2.9}$$

$$31+8\Psi_{0,9}=4e^3\mathcal{Q}+27e\mathcal{Q}^2, \tag{2.10}$$

$$691-16\Psi_{0,11}=16e^4\mathcal{Q}+486e^2\mathcal{Q}^2+189\mathcal{Q}^3, \tag{2.11}$$

$$5461+8\Psi_{0,13}=16e^5\mathcal{Q}+2016e^3\mathcal{Q}^2+3429e\mathcal{Q}^3, \tag{2.12}$$

$$929569-64\Psi_{0,15}=256e^6\mathcal{Q}+130464e^4\mathcal{Q}^2+667872e^2\mathcal{Q}^3+130977\mathcal{Q}^4. \tag{2.13}$$

Table I

Now, we will give a detailed proof of the differential equations (1.15)–(1.17).

Theorem 2.2. *If $\mathcal{P}$ and $\mathcal{Q}$ are defined by* (1.10) *and* (1.12)*, respectively, then $\mathcal{P}$ and $\mathcal{Q}$ satisfy the differential equations* (1.15) *and* (1.17)*, respectively.*

Proof. Recall the identity [13, eq. (17)]

$$\begin{aligned}\Big(\frac{1}{4}\cot\frac{\theta}{2}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\Big)^2=&\Big(\frac{1}{4}\cot\frac{\theta}{2}\Big)^2+\sum_{n=1}^{\infty}\frac{q^n\cos n\theta}{(1-q^n)^2}\\ &+\frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(1-\cos n\theta).\end{aligned} \tag{2.14}$$

Replacing θ by $\pi+\theta$ in (2.14) and expanding $\sin n\theta$ and $\cos n\theta$ in Maclaurin series, we obtain the Taylor series expansion at 0,

$$\begin{aligned}&\Big(\frac{1}{8}\mathcal{P}\theta+\frac{1}{16}\mathcal{Q}\frac{\theta^3}{3!}+\Big(\frac{1}{8}+\Psi_{0,5}\Big)\frac{\theta^5}{5!}+\Big(\frac{17}{32}-\Psi_{0,7}\Big)\frac{\theta^7}{7!}+\cdots\Big)^2\\ &=\frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^n}{(1-q^n)^2}+\frac{1}{2}\Psi_{0,1}\\ &\quad+\Big(\frac{1}{32}+\Psi_{1,2}-\frac{1}{2}\Psi_{0,3}\Big)\frac{\theta^2}{2!}+\Big(\frac{1}{16}-\Psi_{1,4}+\frac{1}{2}\Psi_{0,5}\Big)\frac{\theta^4}{4!}+\cdots.\end{aligned} \tag{2.15}$$

If we compare the coefficient of θ^2 on both sides of (2.15), then we deduce that

$$\mathcal{P}^2 = 1 + 32\Psi_{1,2} - 16\Psi_{0,3} = \mathcal{Q} + 32\Psi_{1,2}. \tag{2.16}$$

By the definition of $\Psi_{0,s}$, $s \geq 1$, it is clear that

$$q\frac{d\Psi_{0,s}}{dq} = \Psi_{1,s+1} \quad \text{and} \quad q\frac{d\Psi_{r,0}}{dq} = \Psi_{r+1,1}. \tag{2.17}$$

So by (2.16), we obtain

$$q\frac{d\mathcal{P}}{dq} = 8q\frac{d\Psi_{0,1}}{dq} = 8\Psi_{1,2} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4}, \tag{2.18}$$

which is the desired identity (1.15).

Similarly, by comparing the coefficients of θ^4 from (2.15), we can find that

$$\frac{\mathcal{P}\mathcal{Q}}{16} = \frac{1}{16} - \Psi_{1,4} + \frac{1}{2}\Psi_{0,5}. \tag{2.19}$$

Hence, from (2.19), we derive that

$$q\frac{d\mathcal{Q}}{dq} = -16q\frac{d\Psi_{0,3}}{dq} = -16\Psi_{1,4} = \mathcal{P}\mathcal{Q} - (1 + 8\Psi_{0,5}) = \mathcal{P}\mathcal{Q} - e\mathcal{Q},$$

from (2.8). □

To find a differential equation for $e(q)$, we need another trigonometric identity proved by Ramamani [12, eq. (1.5)].

Lemma 2.3.

$$\begin{aligned}\Big(\frac{1}{4}\cot\frac{\theta}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\Big)^3 &= \Big(\frac{\cot\theta/2}{4}\Big)^3 - \frac{3}{2}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^3}\sin n\theta \\ &+ \frac{3}{4}\sum_{n=1}^{\infty}\frac{(n+1)q^n}{(1-q^n)^2}\sin n\theta - \frac{1}{16}\sum_{n=1}^{\infty}\frac{(2n^2+1)q^n}{1-q^n}\sin n\theta \\ &+ \frac{3}{8}\cot\frac{\theta}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{3}{2}\Big(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\Big)\Big(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\Big).\end{aligned} \tag{2.20}$$

Theorem 2.4. *If $e(q)$ is defined by* (1.11), *then $e(q)$ satisfies differential equation* (1.16).

Note that for the proof of Theorem 2.4, Ramamani [11, p. 116] briefly mentioned that the equation (1.16) can be obtained by comparing the coefficient of θ in the Taylor expansions around 0 in (2.20), after replacing θ by $\pi + \theta$. We will show this here in detail. We first need the following simple, but useful fact.

Lemma 2.5. *Let $P(q)$, $\mathcal{P}(q)$, and $e(q)$ be as in* (1.1), (1.10), *and* (1.11), *then*

$$P(q) = 3\mathcal{P}(q) - 2e(q). \tag{2.21}$$

Proof. By (2.3), we obtain

$$\begin{aligned}3\mathcal{P}(q)-2e(q)=&3\Big(1+8\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}-8\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\Big)\\&-2\Big(1+24\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\Big)\\=&1-24\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}-24\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\\=&1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.\end{aligned}$$

□

Now to simplify the infinite series on the right hand side of (2.20), a new series representation for $\Psi_{2,1}$ shown below is necessary.

Lemma 2.6. *If $\Psi_{r,s}(q)$ is defined by* (1.9)*, then*

$$\Psi_{2,1}(q):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n(1+q^n)}{(1-q^n)^3}. \tag{2.22}$$

Proof. By the definition of $\Psi_{r,s}(q)$, we easily derive that

$$\Psi_{1,s}(q)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^sq^n}{(1-q^n)^2}. \tag{2.23}$$

Setting $s=0$ in (2.23), we find that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n}{(1-q^n)^2}=\Psi_{1,0}(q).$$

Differentiate both sides of the above equality and then multiply by q. By (2.17), in the case $r=1$, we complete the proof. □

Proof of Theorem 2.4. After replacing θ by $\pi+\theta$ in (2.20) and comparing the coefficients of θ for the Taylor expansions at 0, we can find that

(2.24)

$$\begin{aligned}0=&-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{(1-q^n)^3}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{(1-q^n)^2}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2q^n}{(1-q^n)^2}\\&-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n^3+n)q^n}{1-q^n}+\left(\frac{3}{16}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)\cdot\left(1+8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n}\right).\end{aligned}$$

For convenience, set

$$S_1:=-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{(1-q^n)^3}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{(1-q^n)^2}, \tag{2.25}$$

$$S_2 :=\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2q^n}{(1-q^n)^2}, \tag{2.26}$$

$$S_3 := -\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n^3+n)q^n}{1-q^n}, \tag{2.27}$$

$$S_4 := \left(\frac{3}{16}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)\cdot\left(1+8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n}\right). \tag{2.28}$$

By (2.22) and a simple calculation, we obtain

$$S_1 = -\frac{3}{4}\Psi_{2,1}.$$

For S_2, we have

$$S_2 = \frac{3}{4}q\frac{d}{dq}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n}\right) = \frac{3}{32}\left(q\frac{d\mathcal{P}}{dq}\right) = \frac{3}{128}(\mathcal{P}^2-\mathcal{Q}), \tag{2.29}$$

where the last equality comes from (1.15). By the definitions of $\mathcal{P}$ and $\mathcal{Q}$ given in (1.10) and (1.12), respectively,

$$S_3 = \frac{\mathcal{Q}-\mathcal{P}}{128}.$$

Finally, use (2.21) to deduce that

$$S_4 = \frac{3(1-P)\mathcal{P}}{128} = \frac{(1-3\mathcal{P}-2e)\mathcal{P}}{128}.$$

It follows that

$$\begin{aligned} 0 =& S_1+S_2+S_3+S_4 \\ =& -\frac{3}{4}\Psi_{2,1}+\frac{3(\mathcal{P}^2-\mathcal{Q})}{128}+\frac{\mathcal{Q}-\mathcal{P}}{128}+\frac{\mathcal{P}-3\mathcal{P}^2-2e\mathcal{P}}{128}, \end{aligned}$$

and hence we obtain

$$\Psi_{2,1} = \frac{e\mathcal{P}-\mathcal{Q}}{48}.$$

By using

$$q\frac{de}{dq} = 24\Psi_{2,1},$$

we complete the proof. □

Remark. It is possible to derive (1.15)–(1.17) from the analogous formulas in level one.

Let $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, and use the notations $\mathcal{E}_2$, e_2, and $\mathcal{E}_4$, rather than $\mathcal{P}$, e, and $\mathcal{Q}$, respectively, because we want to focus on their weights. Then the differential equations (1.15)–(1.17) are, in these notations,

$$\Theta\mathcal{E}_2 = \frac{\mathcal{E}_2^2-\mathcal{E}_4}{4}, \tag{2.30}$$

$$\Theta e_2 = \frac{e_2\mathcal{E}_2-\mathcal{E}_4}{2}, \tag{2.31}$$

$$\Theta\mathcal{E}_4 = \mathcal{E}_2\mathcal{E}_4 - e_2\mathcal{E}_4. \tag{2.32}$$

As usual, a holomorphic function on the upper half plane $\mathbb{H}$ and at the cusps of the congruence subgroup Γ is called a modular form of weight k with respect to Γ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. For an even integer $k \geq 2$, let $M_k(\Gamma_0(2))$ denote the space of modular forms of weight k on $\Gamma_0(2)$. Then we know that the operator

$$f \to \Theta f - \frac{k}{12}E_2 f \tag{2.33}$$

maps $M_k(\Gamma_0(2))$ to $M_{k+2}(\Gamma_0(2))$ (see Exercise no. 7 [9, p. 123]). Let $g(z) := E_2(z) - 2E_2(2z)$. Then $g(z) \in M_2(\Gamma_0(2))$ (see [2, Lemma 3.3]). So for any constant α, the operator

$$f \to \Theta f - \frac{k}{12}(E_2 + \alpha g)f \tag{2.34}$$

maps $M_k(\Gamma_0(2))$ to $M_{k+2}(\Gamma_0(2))$. In particular, if we set $\alpha = -2$, and use (2.21), i.e., $\mathcal{E}_2(z) = (4E_2(2z) - E_2(z))/3$, then we have the following lemma.

Lemma 2.7. *Let $f \in M_k(\Gamma_0(2))$, then*

$$\Theta f - \frac{k}{4}\mathcal{E}_2 f \in M_{k+2}(\Gamma_0(2)). \tag{2.35}$$

Now, by (2.35) applied to the modular form $e_2 \in M_2(\Gamma_0(2))$, we have $\Theta e_2 - \frac{\mathcal{E}_2 e_2}{2} \in M_4(\Gamma_0(2))$. By computing the first three terms in the q-expansion (which are enough to exceed the bound coming from the valence formula), we can prove the equality

$$\Theta e_2 - \frac{\mathcal{E}_2 e_2}{2} = -\frac{\mathcal{E}_4}{2},$$

which is exactly (2.31). Similarly, we can derive (2.32), after applying (2.35) to $\mathcal{E}_4 \in M_4(\Gamma_0(2))$.

Since $\mathcal{E}_2$ is not a modular form of weight 2, we cannot use Lemma 2.7 to derive (2.30). So first we prove that $\mathcal{E}_2^2 - 4\Theta\mathcal{E}_2 \in M_4(\Gamma_0(2))$. We need the transformation formula for $\mathcal{E}_2$.

Lemma 2.8. *We have*

$$\mathcal{E}_2\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^2\mathcal{E}_2(z) + \frac{2}{\pi i}c(cz+d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2). \tag{2.36}$$

Proof. Recall the transformation formula [16, p. 68]

$$E_2\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^2 E_2(z) + \frac{6}{\pi i}c(cz+d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{2.37}$$

So for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$E_2\left(2\Big(\frac{az+b}{cz+d}\Big)\right) = E_2\left(\frac{a(2z)+b}{\frac{c}{2}(2z)+d}\right) = (cz+d)^2 E_2(2z) + \frac{3}{\pi i}c(cz+d).$$

Hence, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\begin{aligned}\mathcal{E}_2\Big(\frac{az+b}{cz+d}\Big) =&\frac{4}{3}\Big((cz+d)^2E_2(2z)+\frac{3}{\pi i}c(cz+d)\Big)\\ &-\frac{1}{3}\Big((cz+d)^2E_2(z)+\frac{6}{\pi i}c(cz+d)\Big)\\ =&(cz+d)^2\mathcal{E}_2(z)+\frac{2}{\pi i}c(cz+d).\end{aligned}$$

□

It is clear from (2.36) that

$$\Theta\mathcal{E}_2\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^4\Theta\mathcal{E}_2(z)+\frac{c(cz+d)^3}{\pi i}\mathcal{E}_2(z)-\frac{c^2(cz+d)^2}{\pi^2}. \tag{2.38}$$

So, by (2.36) and (2.38), for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$,

$$\begin{aligned}\mathcal{E}_2^2\Big(\frac{az+b}{cz+d}\Big)-4\Theta\mathcal{E}_2\Big(\frac{az+b}{cz+d}\Big) =&\Big((cz+d)^2\mathcal{E}_2(z)+\frac{2}{\pi i}c(cz+d)\Big)^2\\ &-4\Big((cz+d)^4\Theta\mathcal{E}_2(z)+\frac{c(cz+d)^3}{\pi i}\mathcal{E}_2(z)-\frac{c^2(cz+d)^2}{\pi^2}\Big)\\ =&(cz+d)^4\Big(\mathcal{E}_2^2(z)-4\Theta\mathcal{E}_2(z)\Big).\end{aligned}$$

Hence $\mathcal{E}_2^2 - 4\Theta\mathcal{E}_2 \in M_4(\Gamma_0(2))$. Then by examining at the first three terms in the q-expansions, we find that

$$\mathcal{E}_2^2 - 4\Theta\mathcal{E}_2 = \mathcal{E}_4,$$

which is the desired result (2.30).

3. A differential equation depending on weights of modular forms

The differential equation in the upper half plane $z \in \mathbb{H}$

$$f''(z)-\frac{k+1}{6}E_2(z)f'(z)+\frac{k(k+1)}{12}E_2'(z)f(z)=0, \tag{3.1}$$

was originally studied by M. Kaneko and D. Zagier in [8]. Here the symbol $'$ denotes the Θ-operator $(2\pi i)^{-1}d/dz = q\cdot d/dq$, $(q = e^{2\pi i z})$. For convenience, in this section, we will use this notation. Then it is known, that for $k \equiv 0, 4 \pmod{12}$, there exists a modular solution of (3.1)

$$E_4(z)^{k/4}F\left(-\frac{k}{12},-\frac{k-4}{12};-\frac{k-5}{6};\frac{1728}{j(z)}\right),$$

where

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\quad (a)_n=a(a+1)\cdots(a+n-1),\quad |x|<1,$$

is the Gaussian hypergeometric series, and $j(z)$ is the elliptic modular invariant. Various modular forms on some subgroups were obtained in [6] as solutions to this differential equation, where the groups depend on the choice of k. In particular, on $\Gamma_0(4)$, Ono [10] constructed a family of differential endomorphisms and carried out a similar analysis for modular forms on this group, including those of half-integral weight.

In addition to the modular solutions, quite remarkable was an occurrence of a *quasi–modular form*, not of weight k as in the modular case but of weight $k+1$. Along the same lines, Kaneko and Koike [7] found some examples of quasi–modular forms as solutions to an analogous differential equation attached to the group $\Gamma_0^*(2)$, which are *not* contained in the full modular group, where $\Gamma_0^*(2)$ is defined by

$$\Gamma_0^*(2) = \left\langle \Gamma_0(2), \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \right\rangle,$$

where the modular subgroup $\Gamma_0(2)$ is defined in (1.13).

The differential equation on which we focus in this section is

$$f''(z) - \frac{k+1}{2}\mathcal{E}_2(z)f'(z) + \frac{k(k+1)}{4}\mathcal{E}_2'(z)f(z) = 0, \tag{3.2}$$

where $\mathcal{E}_2(z)$ is a quasi–modular form of weight 2 on $\Gamma_0(2)$ defined by (1.14), i.e.,

$$\mathcal{E}_2(z) = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^n}{1-q^n}.$$

In the present section, for any positive even k, we construct solutions of the differential equation (3.2), which are indeed modular forms of weight k on $\Gamma_0(2)$.

A simple calculation using the differential equations (2.30)–(2.32) shows that $\mathcal{E}_2$ is the logarithmic derivative of the modular form

$$D := \frac{e_2^2 - \mathcal{E}_4}{64} = q + 8q^2 + 28q^3 + 64q^4 + \cdots \tag{3.3}$$

of weight 4 on $\Gamma_0(2)$.

Define a sequence of polynomials $A_n(x)$ by

$$A_0(x) = 1,\ A_1(x) = x,\ A_{n+1}(x) = xA_n(x) + \lambda_n A_{n-1}(x)\ (n = 1, 2, \dots), \tag{3.4}$$

where

$$\lambda_n = -64\frac{(n+1)^2}{(2n+1)(2n+3)}. \tag{3.5}$$

The polynomial $A_n(x)$ is an even or odd polynomial according as n is even or odd, respectively. We also define a second sequence of polynomials $B_n(x)$ by the same recursion with different initial values as follows:

$$B_0(x) = 0,\ B_1(x) = 1,\ B_{n+1}(x) = xB_n(x) + \lambda_n B_{n-1}(x)\ (n = 1, 2, \dots). \tag{3.6}$$

The polynomial $B_n(x)$ has opposite parity, i.e., it is even if n is odd and odd if n is even. Then our first result is given in the following theorem.

Theorem 3.1. *Let $k = 2n+2$ $(n = 0, 1, 2, \dots)$. Then the following modular form of weight k on $\Gamma_0(2)$,*

$$D^{n/2} A_n\Big(\frac{e_2}{\sqrt{D}}\Big)\frac{2e_2}{3} + D^{(n-1)/2} B_n\Big(\frac{e_2}{\sqrt{D}}\Big)\frac{\mathcal{E}_4}{3}, \tag{3.7}$$

is a solution of (3.2), where D is defined by (3.3).

Remark. An element of degree k in the ring $\mathbb{C}[e_2, D]$ is referred to as a modular form of weight k on $\Gamma_0(2)$. Note that $\sqrt{D}$ does not really occur due to the evenness and oddness of $A_n(x)$ and $B_n(x)$ on each n.

Let the operator ϑ_k be denoted by

$$\vartheta_k(f) := f' - \frac{k}{4}\mathcal{E}_2 f, \tag{3.8}$$

which is the formula (2.35) in Lemma 2.7. Using (2.31), we find that

$$\vartheta_2(e_2) = -\frac{\mathcal{E}_4}{2}. \tag{3.9}$$

Similarly by (3.8) and (2.32), we obtain

$$\vartheta_4(\mathcal{E}_4) = -e_2\mathcal{E}_4. \tag{3.10}$$

We also deduce that

$$\vartheta(D) = \frac{\big(\vartheta(e_2^2) - \vartheta(\mathcal{E}_4)\big)}{64} = 0, \tag{3.11}$$

after an application of (3.9) and (3.10).

If f and g have weights k and l, the Leibniz rule

$$\vartheta_{k+l}(fg) = \vartheta_k(f)g + f\vartheta_l(g)$$

holds. We sometimes drop the suffix of the operator ϑ_k when the weights of modular forms we consider are clear. With this ϑ_k operator, the equation (3.2) can be rewritten in the following lemma.

Lemma 3.2. *The differential equation* (3.2) *is equivalent to*

$$\vartheta_{k+2}\vartheta_k(f) = \frac{k(k+2)}{16}\mathcal{E}_4 f. \tag{3.12}$$

Proof. By the definition of the ϑ–operator in (3.8), we obtain

$$\begin{aligned}\vartheta_{k+2}\vartheta_k(f) &= \vartheta_{k+2}\Big(f' - \frac{k}{4}\mathcal{E}_2 f\Big)\\ &= \Big(f' - \frac{k}{4}\mathcal{E}_2 f\Big)' - \frac{k+2}{4}\mathcal{E}_2\Big(f' - \frac{k}{4}\mathcal{E}_2 f\Big)\\ &= -\frac{k(k+2)}{4}\mathcal{E}_2' f + \frac{k(k+2)}{16}\mathcal{E}_2^2 f,\end{aligned}$$

where, in the last equality, we employed the equation (3.2), i.e.

$$f'' - \frac{k+1}{2}\mathcal{E}_2 f' = -\frac{k(k+1)}{4}\mathcal{E}_2' f.$$

So we complete our proof by using (1.15). □

Proof of Theorem 3.1. Let F_k denote the form in (3.7) in Theorem 3.1. We first establish the recurrence relation

$$F_{k+2} = e_2 F_k + \lambda_n D F_{k-2}, \tag{3.13}$$

where $n = (k-2)/2$. This is a consequence of the recurrence relations for A_n and B_n as in (3.4) and (3.6), respectively. Then

$$\begin{aligned}
e_2F_k + \lambda_n DF_{k-2} =& e_2\left(D^{n/2}A_n\left(\frac{e_2}{\sqrt{D}}\right)\frac{2e_2}{3} + D^{(n-1)/2}B_n\left(\frac{e_2}{\sqrt{D}}\right)\frac{\mathcal{E}_4}{3}\right)\\
&+ \lambda_n D\left(D^{(n-1)/2}A_{n-1}\left(\frac{e_2}{\sqrt{D}}\right)\frac{2e_2}{3} + D^{(n-2)/2}B_{n-1}\left(\frac{e_2}{\sqrt{D}}\right)\frac{\mathcal{E}_4}{3}\right)\\
=& D^{(n+1)/2}\left(\frac{e_2}{\sqrt{D}}A_n\left(\frac{e_2}{\sqrt{D}}\right) + \lambda_n A_{n-1}\left(\frac{e_2}{\sqrt{D}}\right)\right)\frac{2e_2}{3}\\
&+ D^{n/2}\left(\frac{e_2}{\sqrt{D}}B_n\left(\frac{e_2}{\sqrt{D}}\right) + \lambda_n B_{n-1}\left(\frac{e_2}{\sqrt{D}}\right)\right)\frac{\mathcal{E}_4}{3}\\
=& D^{(n+1)/2}A_{n+1}\left(\frac{e_2}{\sqrt{D}}\right)\frac{2e_2}{3} + D^{n/2}B_{n+1}\left(\frac{e_2}{\sqrt{D}}\right)\frac{\mathcal{E}_4}{3}\\
=& F_{k+2}.
\end{aligned}$$

Now we prove by induction that F_k satisfies the equation (3.12). For the base step, we first have to check the cases $k = 2$ ($n = 0$) and $k = 4$ ($n = 1$). In the case $k = 2$, from the formula (3.7), we have $F_2 = 2e_2/3$, which is clearly a modular form of weight 2 on $\Gamma_0(2)$. Moreover, using (3.9) and (3.10), we obtain

$$\vartheta^2\left(\frac{2e_2}{3}\right) = \vartheta\left(-\frac{\mathcal{E}_4}{3}\right) = \frac{e_2\mathcal{E}_4}{3} = \frac{2(2+2)}{16}\mathcal{E}_4 \cdot \frac{2e_2}{3},$$

which satisfies the equation (3.12). Similarly, we can deduce that $F_4 = (2e_2^2 + \mathcal{E}_4)/3$ is a modular form of weight 4 satisfying (3.12).

Assume F_{k-2} and F_k satisfy (3.12). Then by using (3.13) and the formulas (3.9), (3.10), and (3.11), we have

$$\begin{aligned}
\vartheta^2(F_{k+2}) =& \vartheta\left(\vartheta(F_k)e_2 - \frac{\mathcal{E}_4F_k}{2}\right) + \lambda_n D\vartheta^2(F_{k-2})\\
=& \vartheta^2(F_k)e_2 + \frac{e_2\mathcal{E}_4F_k}{2} - \mathcal{E}_4\vartheta(F_k) + \lambda_n D\vartheta^2(F_{k-2})\\
=& \frac{k(k+2)}{16}e_2\mathcal{E}_4F_k + \frac{e_2\mathcal{E}_4F_k}{2} - \mathcal{E}_4\vartheta(F_k) + \frac{k(k-2)}{16}\lambda_n D\mathcal{E}_4F_{k-2}\\
=& \frac{k^2+2k+8}{16}e_2\mathcal{E}_4F_k + \frac{k(k-2)}{16}\lambda_n D\mathcal{E}_4F_{k-2} - \mathcal{E}_4\vartheta(F_k).
\end{aligned} \tag{3.14}$$

Hence we find that, by (3.14) and (3.13),

$$\begin{aligned}
\vartheta^2(F_{k+2}) -& \frac{(k+2)(k+4)}{16}\mathcal{E}_4F_{k+2}\\
=& \left(\frac{k^2+2k+8}{16} - \frac{(k+2)(k+4)}{16}\right)e_2\mathcal{E}_4F_k\\
&+ \left(\frac{k(k-2)}{16} - \frac{(k+2)(k+4)}{16}\right)\lambda_n D\mathcal{E}_4F_{k-2} - \mathcal{E}_4\vartheta(F_k)
\end{aligned}$$

$$= -\mathcal{E}_4\left(\frac{k}{4}e_2F_k + \vartheta(F_k) + \frac{k+1}{2}\lambda_n DF_{k-2}\right).$$

To prove the theorem it will therefore suffice to show that

$$\frac{ke_2F_k}{4} + \vartheta(F_k) = -\frac{k+1}{2}\lambda_n DF_{k-2}. \tag{3.15}$$

Again, we will prove the equality (3.15) by induction on k. For the case $k = 4$ ($n = 1$), the equation is checked directly as follows by (3.9) and (3.10):

$$\begin{aligned}\frac{4e_2F_4}{4} + \vartheta(F_4) =& e_2\frac{2e_2^2+\mathcal{E}_4}{3} + \vartheta\left(\frac{2e_2^2+\mathcal{E}_4}{3}\right)\\ =&(e_2^2-\mathcal{E}_4)\frac{2e_2}{3}\\ =&-\frac{5}{2}\left(-64\frac{(1+1)^2}{(2\cdot 1+1)(2\cdot 1+3)}\right)\left(\frac{e_2^2-\mathcal{E}_4}{64}\right)\frac{2e_2}{3}\\ =&-\frac{4+1}{2}\lambda_1 DF_2,\end{aligned}$$

where the last equality comes from the relation (3.5) and (3.3). Using (3.13), we can rewrite F_{k+2} as

$$F_{k+2} = \frac{1}{2(k+1)}((k+2)e_2F_k - 4\vartheta(F_k)). \tag{3.16}$$

Assume that (3.15) is valid for k, i.e., $\vartheta^2(F_k) = 0$. Hence by (3.13) and by applying the ϑ–operator to F_{k+2} in (3.16), we find that

$$\begin{aligned}\frac{k+2}{4}e_2F_{k+2} + \vartheta(F_{k+2}) =& \frac{k+2}{8(k+1)}e_2((k+2)e_2F_k - 4\vartheta(F_k))\\ &+\frac{k+2}{2(k+1)}\left(-\frac{\mathcal{E}_4F_k}{2} + e_2\vartheta(F_k)\right) - \frac{2}{k+1}\vartheta^2(F_k)\\ =&\frac{(k+2)^2}{8(k+1)}(e_2^2-\mathcal{E}_4)F_k\\ =&-\frac{k+3}{2}\lambda_{n+1}DF_k.\end{aligned}$$

Here we have used the induction assumption. Hence the proof of (3.15) is complete, and so then the proof of theorem 3.1 is also complete. □

We next indicate that the solutions of (3.2) have a hypergeometric structure. Let

$$j_2 := \frac{e_2^2}{D} = \frac{1}{q} + 40 + 276q - 2048q^2 + \cdots.$$

Then j_2 is a $\Gamma_0(2)$−invariant function which generates the field of modular functions on $\Gamma_0(2)$ and the normalized function $j_2 - 40$ is often referred to as the "Hauptmodul" for the group $\Gamma_0(2)$.

Theorem 3.3. *For even $k \geq 4$, the differential equation* (3.2) *has solutions which are normalized modular forms of weight k on $\Gamma_0(2)$, a generator of which is given by*

$$e_2^{\frac{k}{2}} F\left(-\frac{k}{4}, -\frac{k-2}{4}; -\frac{k-1}{2}; \frac{64}{j_2}\right). \tag{3.17}$$

Proof. It is sufficient to show that

$$f := \sum_{0 \leq i \leq k/4} \frac{(-\frac{k}{4})_i(-\frac{k-2}{4})_i}{(-\frac{k-1}{2})_i i!} 64^i D^i e_2^{\frac{k}{2}-2i}$$

is a solution of (3.12), since f is a normalized modular form of weight k on $\Gamma_0(2)$. Since

$$\mathcal{E}_4 = e_2^2 - 64D,$$

by (3.8), we find that

$$\vartheta(e_2) = -\frac{\mathcal{E}_4}{2} = 32D - \frac{e_2^2}{2}.$$

Using these, we obtain

$$\vartheta^2(D^i e_2^{\frac{k}{2}-2i}) = \alpha D^i e_2^{\frac{k}{2}-2i+2} + \beta D^{i+1} e_2^{\frac{k}{2}-2i} + \gamma D^{i+2} e_2^{\frac{k}{2}-2i-2}$$

with

$$\alpha = \frac{(k-4i)(k-4i+2)}{16}, \quad \beta = -8(k-4i)^2, \quad \gamma = 256(k-4i)(k-4i-2). \tag{3.18}$$

Hence, for

$$f = \sum_{0 \leq i \leq k/4} a_i D^i e_2^{\frac{k}{2}-2i} \quad \text{with} \quad a_i = 64^i \frac{(-\frac{k}{4})_i(-\frac{k-2}{4})_i}{(-\frac{k-1}{2})_i i!},$$

we have

$$\vartheta^2(f) - \frac{k(k+2)}{16}\mathcal{E}_4 f = \sum_{0 \leq i \leq k/4} a_i' D^{i+1} e_2^{\frac{k}{2}-2i},$$

for some constants a_i'. We can complete the proof by showing that $a_i' = 0$. By the definition of a_i and (3.18), we can express a_i' in terms of a_i, namely,

$$\begin{aligned}
a_i' =& \frac{(k-4i-4)(k-4i-2)}{16} a_{i+1} - 8(k-4i)^2 a_i \\
&+ 256(k-4i+4)(k-4i+2)a_{i-1} - \frac{k(k+2)}{16}(a_{i+1} - 64a_i) \\
=& a_i \times \Bigg\{ -\frac{(k-4i-4)(k-4i-2)(k-4i)(k-4i-2)}{2(k-2i-1)(i+1)} \\
&- 8(k-4i)^2 - 32(k-2i+1)i \\
&- \frac{k(k+1)}{16}\left(-8\frac{(k-4i)(k-4i-2)}{(k-2i-1)(i+1)} - 64\right)\Bigg\} \\
=& 0,
\end{aligned}$$

after a simple algebraic calculation. □

Remark. Solutions of (3.2) can be reformulated in terms of Rankin–Cohen brackets [17]. For modular forms f and g of weights k and l, define a modular form $[f,g]$ of weight $k+l+2$ by

$$[f,g]=kfg'-lf'g$$

("Rankin–Cohen brackets of degree 1"). By the definition of the ϑ–operator in (3.8), the above equation may also be written as

$$[f,g]=kf\vartheta_l(g)-l\vartheta_k(f)g. \tag{3.19}$$

Lemma 3.4. *Suppose F_k satisfies the differential equation* (3.12). *Then*

$$\vartheta([F_k,e_2])=\frac{k-2}{8}[F_k,\mathcal{E}_4] \tag{3.20}$$

and

$$\vartheta((k-2)[F_k,\mathcal{E}_4])+4\vartheta([F_k,e_2^2])=\frac{(k-4)(k+2)}{2}[F_k,e_2]. \tag{3.21}$$

Proof. Since F_k is a solution of (3.12), F_k satisfies the equality

$$\vartheta^2(F_k)=\frac{k(k+2)}{16}\mathcal{E}_4F_k.$$

From this and the use of (3.9) and (3.10), we obtain

$$\begin{aligned}\vartheta([F_k,e_2])=&\vartheta\left(-\frac{k}{2}F_k\mathcal{E}_4-2e_2\vartheta(F_k)\right)\\=&-\frac{k}{2}\vartheta(F_k)\mathcal{E}_4+\frac{k}{2}e_2\mathcal{E}_4F_k-2\left(\frac{k(k+2)}{16}\mathcal{E}_4F_k\right)e_2+\mathcal{E}_4\vartheta(F_k)\\=&-\frac{k(k-2)}{8}e_2\mathcal{E}_4F_k-\frac{k-2}{2}\mathcal{E}_4\vartheta(F_k)\\=&\frac{k-2}{8}[F_k,\mathcal{E}_4],\end{aligned}$$

which proves (3.20). Similarly, by using (3.19), (3.9) and (3.10), we can prove (3.21). □

4. A class of infinite series connected with triangular numbers

On page 188 of his lost notebook [15], Ramanujan examines the series,

$$T_{2k}(q):=1+\sum_{n=1}^{\infty}(-1)^n\{(6n-1)^{2k}q^{n(3n-1)/2}+(6n+1)^{2k}q^{n(3n+1)/2}\},\quad |q|<1. \tag{4.1}$$

Note that the exponents $n(3n\pm1)/2$ are the generalized pentagonal numbers. The series $T_{2k}(q)$, $k=1,2,\ldots$, can be represented in terms of the Eisenstein series $P(q)$, $Q(q)$, and $R(q)$. The proofs of all the formulas on page 188 are given in [3].

If we define the series,

$$\mathcal{T}_{2k}:=\mathcal{T}_{2k}(q)=1+\sum_{n=1}^{\infty}(2n+1)^{2k}q^{n(n+1)/2},\quad |q|<1, \tag{4.2}$$

then we can obtain analogous formulas for $\mathcal{T}_{2k}$ in terms of e, $\mathcal{P}$, and $\mathcal{Q}$. Observe that the exponents $n(n+1)/2$ are the triangular numbers T_n defined by

$$T_n := \frac{n(n+1)}{2}, \quad n \geq 0.$$

Recall that Ramanujan's theta function $\psi(q)$ [1, p. 36, Entry 22 (ii)] is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4.3}$$

where $|q| < 1$, and, for any complex number a, we write $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1})$.

We now state four formulas for $\mathcal{T}_{2k}$.

Theorem 4.1. *If $\mathcal{T}_{2k}$ is defined by* (4.2)*, and $\mathcal{P}$, e, and $\mathcal{Q}$ are defined by* (1.10)–(1.12)*, then*

(i) $$\frac{\mathcal{T}_2(q)}{\psi(q)} = \mathcal{P},$$

(ii) $$\frac{\mathcal{T}_4(q)}{\psi(q)} = 3\mathcal{P}^2 - 2\mathcal{Q},$$

(iii) $$\frac{\mathcal{T}_6(q)}{\psi(q)} = 15\mathcal{P}^3 - 30\mathcal{P}\mathcal{Q} + 16e\mathcal{Q},$$

(iv) $$\frac{\mathcal{T}_8(q)}{\psi(q)} = 105\mathcal{P}^4 - 420\mathcal{P}^2\mathcal{Q} + 448e\mathcal{P}\mathcal{Q} - 128e^2\mathcal{Q} - 4\mathcal{Q}^2.$$

Proof. Important in our proofs is the simple identity

$$(2n+1)^2 = 8\frac{n(n+1)}{2} + 1. \tag{4.4}$$

Observe that, by (4.3),

$$\begin{aligned}
\mathcal{P} =& 1 + 8q\frac{d}{dq}\sum_{n=1}^{\infty}(-1)^n \log(1-q^n) \\
=& 1 + 8q\frac{d}{dq}\log\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\
=& 1 + 8q\frac{\frac{d}{dq}\psi(q)}{\psi(q)}.
\end{aligned}$$

Thus, using (4.4), we find that

$$\begin{aligned}
\psi(q)\mathcal{P} =& \psi(q) + 8q\frac{d}{dq}\left(1 + \sum_{n=1}^{\infty} q^{n(n+1)/2}\right) \\
=& \psi(q) + 8\sum_{n=1}^{\infty}\frac{n(n+1)}{2}q^{n(n+1)/2} \\
=& \psi(q) + \sum_{n=1}^{\infty}(2n+1)^2 q^{n(n+1)/2} - \psi(q) + 1
\end{aligned}$$

$$=\mathcal{T}_2(q). \tag{4.5}$$

This completes the proof of (i).

In the proofs of the remaining identities of Theorem 4.1, in each case, we apply the operator $8q\frac{d}{dq}$ to the preceding identity. In each proof we also use the identities

$$8q\frac{d}{dq}\mathcal{T}_{2k}(q) = \mathcal{T}_{2k+2}(q) - \mathcal{T}_{2k}(q), \tag{4.6}$$

which follows from differentiation and the use of (4.4), and

$$8q\frac{d}{dq}\psi(q) = \mathcal{T}_2(q) - \psi(q), \tag{4.7}$$

which arose in the proof of (4.5).

We now prove (ii). Applying the operator $8q\frac{d}{dq}$ to (4.5) and using (4.6) and (4.7), we deduce that

$$\mathcal{P}(q)(\mathcal{T}_2 - \psi(q)) + \psi(q)8q\frac{d\mathcal{P}(q)}{dq} = \mathcal{T}_4(q) - \mathcal{T}_2(q).$$

Employing (i) to simplify and using (1.15), we arrive at

$$\mathcal{T}_4(q) = (3\mathcal{P}^2 - 2\mathcal{Q})\psi(q), \tag{4.8}$$

as desired.

To prove (iii), we apply the operator $8q\frac{d}{dq}$ to (4.8) and use (4.6) and (4.7) to deduce that

$$\begin{aligned}\mathcal{T}_6 - \mathcal{T}_4 =& 8\Big(6\mathcal{P}q\frac{d\mathcal{P}}{dq} - 2q\frac{d\mathcal{Q}}{dq}\Big)\psi(q) + (3\mathcal{P}^2 - 2\mathcal{Q})(\mathcal{T}_2 - \psi(q))\\ =& \Big(12\mathcal{P}(\mathcal{P}^2 - \mathcal{Q}) - 16(\mathcal{P}\mathcal{Q} - e\mathcal{Q})\Big)\psi(q) + (3\mathcal{P}^2 - 2\mathcal{Q})(\mathcal{P} - 1)\psi(q),\end{aligned}$$

where we used (1.15), (1.17) and (i). If we now employ (4.8) and simplify, we obtain (iii).

In general, by applying the operator $8q\frac{d}{dq}$ to $\mathcal{T}_{2k}$ and using (4.6) and (4.7), we find that

$$\mathcal{T}_{2k+2} - \mathcal{T}_{2k} = 8q\frac{d}{dq}g_{2k}(\mathcal{P}, e, \mathcal{Q})\psi(q) + \mathcal{P}g_{2k}(\mathcal{P}, e, \mathcal{Q}),$$

where we define the polynomials $g_{2k}(\mathcal{P}, e, \mathcal{Q})$, $k \geq 1$, by

$$g_{2k}(\mathcal{P}, e, \mathcal{Q}) := \frac{\mathcal{T}_{2k}(q)}{\psi(q)}. \tag{4.9}$$

Then proceeding by induction while using the formula (4.9) for $\mathcal{T}_{2k}$, we find that

$$g_{2k+2}(\mathcal{P}, e, \mathcal{Q}) = 8q\frac{d}{dq}g_{2k}(\mathcal{P}, e, \mathcal{Q}) + \mathcal{P}g_{2k}(\mathcal{P}, e, \mathcal{Q}). \tag{4.10}$$

With the use of (4.10) and the differential equations (1.15)–(1.17), it should now be clear how to prove the remaining identity (iv), and so we omit further details. □

Remark. Observe from Theorem 4.1 that a general formula for $g_{2k}(\mathcal{P}, e, \mathcal{Q})$ contains all products $\mathcal{P}^l e^m \mathcal{Q}^n$, such that $2l + 2m + 4n = 2k$. It seems to be extremely difficult to find a general formula for $g_{2k}(\mathcal{P}, e, \mathcal{Q})$ that would give explicit representations for each coefficient of $\mathcal{P}^l e^m \mathcal{Q}^n$.

5. A combinatorial identity

Let $\widetilde{\sigma}_s$ be defined for $s, n \in \mathbb{N}$, by

$$\widetilde{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s, \tag{5.1}$$

where $\widetilde{\sigma}_1(n) = \widetilde{\sigma}(n)$, and $\widetilde{\sigma}_s(n) = 0$ if $n \notin \mathbb{N}$. Glaisher [5] defined seven quantities which depend on the divisors of n, including (5.1), and found expressions for them in terms of the $\sigma_s(n)$ defined as (1.5). For instance [5],

$$\widetilde{\sigma}_s(n) = \sigma_s(n) - 2^{s+1}\sigma_s(n/2). \tag{5.2}$$

Then the first formula (i) in Theorem 4.1 has an interesting arithmetical interpretation.

Theorem 5.1. *Define* $\widetilde{\sigma}(0) = \frac{1}{8}$. *Then*

$$8 \sum_{\substack{j+k(k+1)/2=n \\ j,k\geq 0}} \widetilde{\sigma}(j) = \begin{cases} (2r+1)^2, & \text{if } n = r(r+1)/2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. By expanding the summands of $\mathcal{P}$ in (1.10) in geometric series and collecting the coefficients of q^n for each positive integer n, we find that

$$\mathcal{P}(q) = 1 + 8\sum_{n=1}^{\infty} \widetilde{\sigma}(n) q^n = 8 \sum_{n=0}^{\infty} \widetilde{\sigma}(n) q^n,$$

upon using the definition $\widetilde{\sigma}(0) = \frac{1}{8}$. Thus, by (4.3) and Theorem 4.1, (i) can be written in the form

$$\left(8\sum_{j=0}^{\infty} \widetilde{\sigma}(j) q^j\right) \cdot \left(\sum_{k=0}^{\infty} q^{k(k+1)/2}\right) = 1 + \sum_{n=1}^{\infty} (2n+1)^2 q^{n(n+1)/2}. \tag{5.4}$$

Equating coefficients of q^n, $n \geq 1$, on both sides of (5.4), we complete the proof. □

Let $\mathbb{N}$ be the set of positive integers. Define

$$\mathcal{A} := \{(x, y) \in \mathbb{N}^2 : 2x^2 + y^2 = 8n + 1, y \text{ is odd}, 2|x\} \tag{5.5}$$

and

$$\mathcal{B} := \{(x, y) \in \mathbb{N}^2 : x^2 + y^2 = 8n + 1, y \text{ is odd}, 4|x\}. \tag{5.6}$$

Then we derive the following combinatorial corollary of Theorem 5.1.

Corollary 5.2. *The number of elements of* $\mathcal{A}$ *and the number of elements of* $\mathcal{B}$ *have the same parity in all cases except when* $n = r(r+1)/2$ *and* $r \equiv 1, 2 \pmod 4$.

Proof. Since

$$\widetilde{\sigma}(j) \equiv \begin{cases} 1 \pmod 2, & \text{if } j = m^2 \text{ or } j = 2m^2, \\ 0 \pmod 2, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{\substack{j+k(k+1)/2=n \\ j,k\geq 0}} \widetilde{\sigma}(j) \equiv \sum_{\substack{j+k(k+1)/2=n \\ j\geq 1,k\geq 0 \\ j=m^2 \text{ or } j=2m^2}} \widetilde{\sigma}(j) \pmod 2. \tag{5.7}$$

After changing variables, it is easy to see that $\mathcal{A}$ and $\mathcal{B}$ can be rewritten as

$$\mathcal{A} = \{(j,k) \mid j > 0, k \geq 0, j + k(k+1)/2 = n, j = m^2 \}$$

and

$$\mathcal{B} = \{ (j,k) \mid j > 0, k \geq 0, j + k(k+1)/2 = n, j = 2m^2\}.$$

Therefore, by (5.7), we find that

$$\begin{aligned} \#\mathcal{A} + \#\mathcal{B} &= \sum_{\substack{j+k(k+1)/2=n \\ j\geq 1,k\geq 0 \\ j=m^2 \text{or} j=2m^2}} 1 \\ &\equiv \sum_{\substack{j+k(k+1)/2=n \\ j\geq 1,k\geq 0 \\ j=m^2 \text{or} j=2m^2}} \widetilde{\sigma}(j) \pmod 2 \\ &\equiv \sum_{\substack{j+k(k+1)/2=n \\ j\geq 1,k\geq 0}} \widetilde{\sigma}(j) \pmod 2 \\ &= \begin{cases} \frac{(2r+1)^2-1}{8}, & \text{if } n = \frac{r(r+1)}{2}, \\ 0, & \text{otherwise} \end{cases} \\ &\equiv \begin{cases} 1 \pmod 2, & \text{when } n = r(r+1)/2 \text{ and } r \equiv 1,\ 2 \pmod 4, \\ 0 \pmod 2, & \text{otherwise.} \end{cases} \end{aligned}$$

So we conclude the result. □

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